

$U(3)$ chiral perturbation theory with infrared regularization¹

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Abstract

We include the η' in chiral perturbation theory without employing $1/N_c$ counting rules. The method is illustrated by calculating the masses and decay constants of the Goldstone boson octet (π, K, η) and the singlet η' up to one-loop order. The effective Lagrangian describing the interactions of the η' with the Goldstone boson octet is presented up to fourth chiral order and the loop integrals are evaluated using infrared regularization, which preserves Lorentz and chiral symmetry.

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1 Introduction

Chiral perturbation theory is the effective field theory of QCD at low energies. The QCD Lagrangian with massless quarks exhibits an $SU(3)_R \times SU(3)_L$ chiral symmetry which is broken down spontaneously to $SU(3)_V$, giving rise to a Goldstone boson octet of pseudoscalar mesons: pions, kaons and the η , which become massless in the chiral limit of zero quark masses. The axial $U(1)$ anomaly, on the other hand, prevents the corresponding singlet state, the η' , from becoming massless even in the chiral limit. Therefore, in conventional chiral perturbation theory the η' is not included explicitly, although it does show up in the form of a contribution to a coupling coefficient of the Lagrangian, a so-called low-energy constant (LEC).

In the large N_c limit the quark loop which is responsible for the $U(1)$ anomaly is suppressed. The chiral symmetry of the QCD Lagrangian is thus extended to $U(3)_R \times U(3)_L$. At the level of the effective theory, the octet of Goldstone bosons converts into a nonet with η' being the ninth member. The properties of this particle are subject to similar constraints as the original Goldstone bosons (π, K, η) and the corresponding effective Lagrangian can be constructed. A systematic expansion of the Green functions of chiral perturbation theory in powers of momenta, quark masses and $1/N_c$ was introduced in [1] and has recently been more firmly established in [2, 3] and [4]. An investigation of these papers reveals that in order to construct the effective Lagrangian including the singlet field no large N_c arguments are necessary. An additional $1/N_c$ counting scheme is imposed only to ensure that loops with an η' are suppressed by powers of $1/N_c$. In particular, the mass of the η' which introduces an additional low energy scale of about 1 GeV is proportional to $1/N_c$ and can therefore be treated perturbatively.

Without invoking large N_c arguments the inclusion of the η' in baryon chiral perturbation theory has been outlined in [5]. This work suffers, however, from the fact that at one loop order the η' loop contributions are substantial due to the large mass of the η' , $m_{\eta'}$. A systematic expansion in the meson masses is possible but its convergence is doubtful, since $m_{\eta'}$ is close to the scale of chiral symmetry breaking $\Lambda_\chi = 4\pi F_\pi \sim 1.2$ GeV with $F_\pi \approx 93$ MeV the pion decay constant. This problem arises for every particle in the effective field theory which has been included explicitly but has a mass similar to or larger than Λ_χ . The inclusion of the lowest lying baryon octet (N, Λ, Σ, Ξ), e.g., spoils the chiral counting scheme if one employs the relativistic Lagrangian and regularizes loop integrals dimensionally. This can be prevented by going to the nonrelativistic limit and treating the baryons as heavy static sources. Within the so-called heavy baryon chiral perturbation theory one obtains a one-to-one correspondence between the number of loops and the chiral order, i.e. a chiral counting scheme emerges [6]. More recently an alternative way of treating massive fields has been proposed in [7] and put on a more solid basis in [8]. The authors keep the relativistic formulation of the Lagrangian but modify the loop integrals in a chiral invariant way. In [8]

this was achieved by employing a modified regularization scheme, the so-called infrared regularization, in which Lorentz and chiral invariance are kept at all stages.

The purpose of the present work is to implement infrared regularization in an effective theory including the η' , while not using any large N_c arguments. In this introductory presentation, we will restrict ourselves to the purely mesonic case and calculate the masses and decay constants of the pseudoscalar mesons. We start in the following section by presenting the effective Lagrangian. The dependence of the LECs on the renormalization scale in QCD is discussed in Sec. 3 and App. A. Section 4 is a presentation of our results for the masses and decay constants up to one-loop order using infrared regularization.

2 The effective Lagrangian

The full effective $U(3)_R \times U(3)_L$ Lagrangian up to fourth chiral order, i.e. including terms up to four derivatives and quadratic in the quark masses, has already been given in [4]. We will therefore restrict ourselves to a repetition of the construction principles which will make it obvious that no $1/N_c$ arguments are required. To this end, consider the QCD Lagrangian in the presence of external sources

$$\mathcal{L}_{QCD} = \mathcal{L}_{QCD}^0 + \bar{q}\gamma_\mu(v^\mu + \gamma_5 a^\mu)q - \bar{q}(s - i\gamma_5 p)q - \frac{g^2}{16\pi^2}\theta(x)\text{tr}_c(G_{\mu\nu}\tilde{G}^{\mu\nu}) \quad (1)$$

with $\tilde{G}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}$ and tr_c is the trace over the color indices. The term \mathcal{L}_{QCD}^0 describes the limit where the masses of the three light quarks and the vacuum angle are set to zero and the external sources $v_\mu(x), a_\mu(x), s(x), p(x)$ are hermitian 3×3 matrices in flavor space. The mass matrix of the three light quarks is contained in the external field s . Under $U(1)_R \times U(1)_L$ the axial $U(1)$ anomaly adds a term $-(g^2/16\pi^2)2N_f\alpha\text{tr}_c(G_{\mu\nu}\tilde{G}^{\mu\nu})$ to the QCD Lagrangian, with N_f being the number of different quark flavors and α the angle of the axial $U(1)$ rotation. The vacuum angle $\theta(x)$ is in this context treated as an external field that transforms under an axial $U(1)$ rotation as

$$\theta(x) \rightarrow \theta'(x) = \theta(x) - 2N_f\alpha. \quad (2)$$

Then the term generated by the anomaly in the fermion determinant is compensated by the shift in the θ source and the Lagrangian from Eq. (1) remains invariant under axial $U(1)$ transformations. The original symmetry group $SU(3)_R \times SU(3)_L$ of the Lagrangian \mathcal{L}_{QCD} is extended to $G = U(3)_R \times U(3)_L$.⁴

⁴To be more precise, the Lagrangian changes by a total derivative which gives rise to the Wess-Zumino term. We will neglect this contribution since the corresponding terms involve five or more meson fields which do not play any role for the discussions here.

This property remains at the level of an effective theory and the additional source θ also shows up in the effective Lagrangian. We assume that this extended symmetry G is spontaneously broken down to $H = U(3)_V$. The nine parameters of the coset space $G/H = U(3)$ correspond then to the lowest lying nonet of pseudoscalar mesons: pions, kaons, η and η' . They can be most conveniently summarized in a matrix valued field

$$U(\phi, \psi) = u^2(\phi, \psi) = \exp\{2i\phi/F + i\psi/3\}, \quad (3)$$

where F is the decay constant of the Goldstone boson octet ϕ in the chiral limit. The unimodular part of the field $U(x)$ contains the degrees of freedom of the octet ϕ

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}, \quad (4)$$

while the phase $\det U(x) = e^{i\psi}$ describes the singlet η_0 . The effective Lagrangian is formed with the fields $U(x)$, derivatives thereof and also includes the external fields: $\mathcal{L}_{\text{eff}}(U, \partial U, \dots, v, a, s, p, \theta)$. Under $U(3)_R \times U(3)_L$ the fields transform as follows

$$\begin{aligned} U' &= RUL^\dagger, & s' + ip' &= R(s + ip)L^\dagger, \\ r'_\mu &= Rr_\mu R^\dagger + iR\partial_\mu R^\dagger, & l'_\mu &= Ll_\mu L^\dagger + iL\partial_\mu L^\dagger, \\ \theta' &= \theta + i \ln \det R - i \ln \det L, \end{aligned} \quad (5)$$

with $r_\mu = v_\mu + a_\mu$, $l_\mu = v_\mu - a_\mu$ and $R \in U(3)_R$, $L \in U(3)_L$, but the Lagrangian remains invariant. The phase of the determinant $\det U(x) = e^{i\psi}$ transforms under axial $U(1)$ as $\psi' = \psi + 2N_f\alpha$ so that the combination $\psi + \theta$ remains invariant. It is more convenient to replace the variable θ in the effective Lagrangian by this invariant combination $\bar{\psi} = \psi + \theta$, $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \dots, v, a, s, p, \bar{\psi})$. One can now construct the effective Lagrangian in these fields that respects the symmetries of the underlying theory. In particular, the Lagrangian is invariant under $U(3)_R \times U(3)_L$ rotations of U and the external fields at a fixed value of the last argument. The most general Lagrangian up to and including terms with two derivatives and one factor of the quark mass matrix reads

$$\begin{aligned} \mathcal{L}^{(0+2)} &= -V_0 + V_1 \langle \nabla_\mu U^\dagger \nabla^\mu U \rangle + V_2 \langle U \chi^\dagger + U^\dagger \chi \rangle + iV_3 \langle U \chi^\dagger - U^\dagger \chi \rangle \\ &\quad + V_4 \langle U^\dagger \nabla_\mu U \rangle \langle U^\dagger \nabla^\mu U \rangle + iV_5 \langle U^\dagger \nabla_\mu U \rangle \nabla^\mu \theta + V_6 \nabla_\mu \theta \nabla^\mu \theta. \end{aligned} \quad (6)$$

The expression $\langle \dots \rangle$ denotes the trace in flavor space and the quark mass matrix $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$ enters in the combination

$$\chi = 2B(s + ip) = 2B\mathcal{M} \quad (7)$$

with $B = -\langle 0|\bar{q}q|0\rangle/F^2$ the order parameter of the spontaneous symmetry violation. The covariant derivatives are defined by

$$\begin{aligned}\nabla_\mu U &= \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu) \\ \nabla_\mu \theta &= \partial_\mu \theta + 2\langle a_\mu \rangle.\end{aligned}\quad (8)$$

The coefficients V_i are functions of the variable $\bar{\psi}$, $V_i(\bar{\psi})$, and can be expanded in terms of this variable. At a given order of derivatives of the meson fields U and insertions of the quark mass matrix \mathcal{M} one obtains an infinite string of increasing powers of $\bar{\psi}$ with couplings which are not fixed by chiral symmetry. Parity conservation implies that the V_i are all even functions of $\bar{\psi}$ except V_3 , which is odd, and $V_1(0) = V_2(0) = F^2/4$ gives the correct normalization for the quadratic terms of the Goldstone boson octet. A transformation of the type $U \rightarrow e^{if(\bar{\psi})}U$ leaves the structure of the effective Lagrangian invariant, but modifies the potentials V_i [3]. We will remove the term $i\langle U^\dagger \nabla_\mu U \rangle \nabla^\mu \theta$ by choosing $f(\bar{\psi})$ accordingly. Alternatively, one could simplify the potential $V_0(\bar{\psi})$ so that it reads $V_0(\bar{\psi}) = v_0^{(0)} + v_0^{(2)}\bar{\psi}^2$. This is achieved by replacing $\bar{\psi} \rightarrow \bar{\psi}e^{g(\bar{\psi})}$ while keeping θ fixed. The function g can be chosen in such a way that it cancels the terms with four and more powers of $\bar{\psi}$ in the expansion of $V_0(\bar{\psi})$. On the other hand, the transformations for U and $\bar{\psi}$ are related to each other via $\psi = -i \ln \det U$, so that one cannot eliminate V_5 and simplify V_0 simultaneously. We prefer working with the Lagrangian in which the potential V_5 has been transformed away and keep V_0 . Note also that V_6 does not contribute to the processes we are considering here and will be neglected.

At fourth chiral order many more terms contribute [4], and we will only present the relevant ones for our present investigation

$$\begin{aligned}\mathcal{L}^{(4)} = & \beta_4 \langle \nabla_\mu U^\dagger \nabla^\mu U \rangle \langle U^\dagger \chi + \chi^\dagger U \rangle + \beta_5 \langle \nabla_\mu U^\dagger \nabla^\mu U (U^\dagger \chi + \chi^\dagger U) \rangle \\ & + \beta_6 \langle U^\dagger \chi + \chi^\dagger U \rangle^2 + \beta_7 \langle U^\dagger \chi - \chi^\dagger U \rangle^2 \\ & + \beta_8 \langle U^\dagger \chi U^\dagger \chi + \chi^\dagger U \chi^\dagger U \rangle + \beta_{12} \langle \chi^\dagger \chi \rangle \\ & + \beta_{17} \langle U^\dagger \nabla_\mu U \rangle \langle U^\dagger \nabla^\mu U \rangle \langle U^\dagger \chi + \chi^\dagger U \rangle \\ & + \beta_{18} \langle U^\dagger \nabla_\mu U \rangle \langle \nabla^\mu U^\dagger \chi - \chi^\dagger \nabla^\mu U \rangle \\ & + i\beta_{25} \langle U^\dagger \chi U^\dagger \chi - \chi^\dagger U \chi^\dagger U \rangle + i\beta_{26} (\langle U^\dagger \chi \rangle^2 - \langle \chi^\dagger U \rangle^2) \\ & - \beta_{52} \partial_\mu \nabla^\mu \theta \langle U^\dagger \chi + \chi^\dagger U \rangle\end{aligned}\quad (9)$$

The operators O_{46} , O_{47} and O_{53} from [4] are not shown here, since they can be removed via the transformation $U \rightarrow U \exp[f_1(\bar{\psi})U^\dagger \nabla^\mu U \nabla_\mu \theta + if_2(\bar{\psi})\partial_\mu \nabla^\mu \theta + f_3(\bar{\psi})\langle U^\dagger \nabla_\mu U \rangle \nabla^\mu \theta]$ [3]. We make use of this possibility and neglect these terms. Furthermore, the vacuum angle has served its purpose in constructing the effective Lagrangian and will be omitted for the processes under consideration. However, one must keep the singlet component of a_μ , $\langle a_\mu \rangle$, in the covariant derivative of $\nabla_\mu \theta = \partial_\mu \theta + 2\langle a_\mu \rangle$.

3 Identifying the singlet field

In the last section we have mentioned that the singlet field η_0 is described by the phase $\det U = e^{i\psi}$. It remains to be seen, however, which choice of η_0 is sensible and how this choice is related to ψ . To this end, consider the dependence of the LECs in the effective Lagrangian on the renormalization scale of QCD. This has been described in detail in [3]. We will therefore just repeat the basic formulae and restrict ourselves to the renormalization of the singlet axial current. The renormalization of the scalar and pseudoscalar operators as well as the one for the topological charge density $\omega = (g^2/16\pi^2) \text{tr}_c(G_{\mu\nu}\tilde{G}^{\mu\nu})$ is not relevant for our purposes and will be neglected. The matrix elements of the singlet axial current $A_\mu^0 = \frac{1}{2}\bar{q}\gamma_\mu\gamma_5 q$ depend on the renormalization scale of QCD since this operator carries anomalous dimension [9]. This operator receives multiplicative renormalization and, therefore, the decay constants associated with the singlet quark current depend on the scale

$$A_\mu^0 \rightarrow Z_A A_\mu^0, \quad F_P^0 \rightarrow Z_A F_P^0, \quad P = \eta, \eta' \quad (10)$$

where the decay constants are given by

$$\langle 0 | A_\mu^0 | P \rangle = i p_\mu F_P^0. \quad (11)$$

The renormalization factor Z_A depends on the running scale of QCD

$$\mu_{QCD} \frac{dZ_A}{d\mu_{QCD}} = \gamma_A Z_A, \quad \gamma_A = -\frac{6N_f(N_c^2 - 1)}{N_c} \left(\frac{g}{4\pi}\right)^4 + \mathcal{O}(g^6) \quad (12)$$

with g being the QCD coupling constant. Note that we work in the isospin limit $\hat{m} = m_u = m_d$, in which $F_{\pi^0}^0$ vanishes. It is convenient to compensate the scale dependence of the singlet axial current by treating the corresponding external source as scale dependent field, so that the effective action of QCD given by the Lagrangian in Eq. (1) becomes scale independent. For $\theta = 0$ the effective action remains the same if we replace the singlet component of the axial field by

$$\langle a_\mu \rangle \rightarrow Z_A^{-1} \langle a_\mu \rangle \quad (13)$$

while the octet component $\hat{a}_\mu = a_\mu - \langle a_\mu \rangle / 3$ is unaffected by a change in the renormalization scale. Thus, if the vacuum angle θ is turned off, the situation for $\langle a_\mu \rangle$ is analogous to the case of the scalar and pseudoscalar external currents in standard chiral perturbation theory, where one introduces the multiplicative constant B , cf. Eq. (7). However, for a finite vacuum angle θ , $\langle a_\mu \rangle$ is subject to an inhomogeneous renormalization [3]. This complication does not arise here since we work with a vanishing vacuum angle in the present investigation. The scale invariance of the Lagrangian translates into the effective theory as follows. For $\theta = 0$ the singlet component of the external axial field appears due to axial

$U(1)$ invariance only in the combination $\nabla_\mu \psi = \partial_\mu \psi - 2\langle a_\mu \rangle$ which acquires multiplicative renormalization

$$\nabla_\mu \psi \rightarrow Z_A^{-1} \nabla_\mu \psi, \quad \text{i.e. } \psi \rightarrow Z_A^{-1} \psi. \quad (14)$$

For the effective Lagrangian to remain invariant under a change of the renormalization scale in QCD, the potentials V_i and β_i must transform accordingly. We will illustrate this method in this section by restricting ourselves to the Lagrangian $\mathcal{L}^{(0+2)}$ from Eq. (6). The pertinent formulae for $\mathcal{L}^{(4)}$ can be found in App. A.

Decomposing the matrix valued field U into its unimodular part and its phase

$$U = e^{\frac{i}{3}\psi} \hat{U} \quad (15)$$

and the axial-vector field a_μ into octet and singlet components

$$a_\mu = \hat{a}_\mu + \frac{1}{3}\langle a_\mu \rangle, \quad (16)$$

the Lagrangian $\mathcal{L}^{(0+2)}$ can be rewritten as ($V_5 = V_6 = 0$),

$$\begin{aligned} \mathcal{L}^{(0+2)} = & -V_0 + V_1 \langle \hat{\nabla}_\mu \hat{U}^\dagger \hat{\nabla}^\mu \hat{U} \rangle + [V_2 + iV_3] e^{\frac{i}{3}\psi} \langle \hat{U} \chi^\dagger \rangle \\ & + [V_2 - iV_3] e^{-\frac{i}{3}\psi} \langle \hat{U}^\dagger \chi \rangle + [\frac{1}{3}V_1 - V_4] \nabla_\mu \psi \nabla^\mu \psi. \end{aligned} \quad (17)$$

The covariant derivative of \hat{U} is given by

$$\hat{\nabla}_\mu \hat{U} = \partial_\mu \hat{U} - i(v_\mu + \hat{a}_\mu) \hat{U} + i\hat{U}(v_\mu - \hat{a}_\mu). \quad (18)$$

For the Lagrangian to remain invariant, the potentials V_i must transform as

$$\begin{aligned} V_0(x) &\rightarrow V_0(Z_A x) \\ V_1(x) &\rightarrow V_1(Z_A x) \\ (V_2 + iV_3)(x) &\rightarrow (V_2 + iV_3)(Z_A x) e^{\frac{i}{3}(Z_A - 1)x} \\ V_4(x) &\rightarrow Z_A^2 V_4(Z_A x) + \frac{1}{3}(1 - Z_A^2)V_1(Z_A x). \end{aligned} \quad (19)$$

These transformation properties of the potentials have consequences for the choice of the singlet field. Consider the free kinetic term for ψ in Eq. (17)

$$[\frac{1}{3}V_1(0) - V_4(0)] \partial_\mu \psi \partial^\mu \psi. \quad (20)$$

Both the coefficient $V_1(0)/3 - V_4(0)$ and ψ are scale dependent quantities with

$$\frac{1}{3}V_1(0) - V_4(0) = \frac{F^2}{12} - V_4(0) \rightarrow Z_A^2 \left(\frac{F^2}{12} - V_4(0) \right). \quad (21)$$

Note that F , the pion decay constant in the chiral limit, does not depend on the running scale of QCD. We prefer to work with a scale independent singlet field η_0 which has the same kinetic term as the octet fields. This is achieved by replacing ψ by η_0 in the effective Lagrangian with

$$\eta_0 = \sqrt{\lambda} \psi \equiv \sqrt{\frac{F^2}{6} - 2V_4(0)} \psi. \quad (22)$$

The potentials are then rescaled according to

$$\begin{aligned} \bar{V}_0(x) &= V_0\left(\frac{F}{\sqrt{\lambda}}x\right) \\ \bar{V}_1(x) &= V_1\left(\frac{F}{\sqrt{\lambda}}x\right) \\ (\bar{V}_2 + i\bar{V}_3)(x) &= (V_2 + iV_3)\left(\frac{F}{\sqrt{\lambda}}x\right) e^{\frac{i}{3}(F-\sqrt{6\lambda})x/\sqrt{\lambda}} \\ \left(\frac{1}{3}\bar{V}_1 - \bar{V}_4\right)(x) &= \frac{F^2}{6\lambda}\left(\frac{1}{3}V_1 - V_4\right)\left(\frac{F}{\sqrt{\lambda}}x\right). \end{aligned} \quad (23)$$

The \bar{V}_i are functions of the singlet field η_0 , $\bar{V}_i(\eta_0/F)$, and do not depend on the renormalization scale μ_{QCD} . On the other hand, $\nabla_\mu \psi$ transforms into

$$\nabla_\mu \psi \rightarrow \frac{1}{\sqrt{\lambda}} \nabla_\mu \eta_0 = \frac{1}{\sqrt{\lambda}} \left(\partial_\mu \eta_0 - 2\sqrt{\lambda} \langle a_\mu \rangle \right) \quad (24)$$

so that

$$\left(\frac{1}{3}V_1 - V_4\right)(\psi) \nabla_\mu \psi \nabla^\mu \psi \rightarrow \frac{6}{F^2} \left(\frac{1}{3}\bar{V}_1 - \bar{V}_4\right) \left(\frac{\eta_0}{F}\right) \nabla_\mu \eta_0 \nabla^\mu \eta_0. \quad (25)$$

Note that the kinetic term for η_0 which is the first term in the expansion of $\bar{V}_1/3 - \bar{V}_4$ is normalized in such a way that $\bar{V}_1(0)/3 - \bar{V}_4(0) = F^2/12$, i.e. $\bar{V}_4(0) = 0$. In the effective theory the scale dependence of the axial current manifests itself in the prefactor $\sqrt{\lambda}$ of $\langle a_\mu \rangle$. We also would like to point out that the quantity $\lambda = F^2/6 - 2V_4(0)$ is indeed a positive number. This can be seen as follows. The pieces of the Lagrangian in Eq. (17) quadratic in the field ψ can be written in the chiral limit as

$$\frac{1}{2} \left[\frac{F^2}{6} - 2V_4(0) \right] \partial_\mu \psi \partial^\mu \psi - v_0^{(2)} \psi^2 \quad (26)$$

where the first and second term constitute the kinetic energy and the mass term, respectively. At lowest order in $1/N_c$ the equality $2v_0^{(2)} = \tau_{GD}$ holds, with τ_{GD} being the topological susceptibility of gluodynamics [3]. The domain of validity for standard $SU(3)$ chiral perturbation theory is restricted by the condition [10]

$$m_s |\langle 0 | \bar{u}u | 0 \rangle| \ll 9\tau_{GD}. \quad (27)$$

Assuming small $1/N_c$ corrections, $v_0^{(2)}$ is thus a positive number. For the free effective Lagrangian in Eq. (26) to make sense, we obtain the constraint

$$\frac{F^2}{6} - 2V_4(0) > 0. \quad (28)$$

Otherwise, the corresponding ψ propagator does not develop a pole and no singlet particle occurs.

We will now apply the Lagrangian with the octet ϕ and η_0 to calculate both the meson masses at lowest order and η_0 - η_8 mixing, which yields the physical states η and η' . The potentials \bar{V}_i are expanded in the singlet field η_0

$$\begin{aligned} \bar{V}_i\left(\frac{\eta_0}{F}\right) &= \bar{v}_i^{(0)} + \bar{v}_i^{(2)} \frac{\eta_0^2}{F^2} + \bar{v}_i^{(4)} \frac{\eta_0^4}{F^4} + \dots \quad \text{for } i = 0, 1, 2, 4 \\ \bar{V}_3\left(\frac{\eta_0}{F}\right) &= \bar{v}_3^{(1)} \frac{\eta_0}{F} + \bar{v}_3^{(3)} \frac{\eta_0^3}{F^3} + \dots . \end{aligned} \quad (29)$$

The expansion coefficients $\bar{v}_i^{(j)}$ are independent of the running scale of QCD, whereas in general the corresponding coefficients $v_i^{(j)}$ of the potentials V_i are not. One observes terms quadratic in the meson fields that contain the factor $\eta_0\eta_8$. Such terms arise from the explicitly chiral symmetry breaking operators $(\bar{V}_2 + i\bar{V}_3)e^{i\sqrt{6}\eta_0/(3F)}\langle\hat{U}\chi^\dagger\rangle + h.c.$ and read

$$-\frac{8}{\sqrt{3}F^2} \left[\frac{F^2}{2\sqrt{6}} + \bar{v}_3^{(1)} \right] B(\hat{m} - m_s)\eta_0\eta_8 \equiv -m_{08}^2\eta_0\eta_8. \quad (30)$$

The states η_0 and η_8 are not mass eigenstates and the mass matrix can be diagonalized by introducing the eigenstates η and η'

$$\begin{aligned} |\eta\rangle &= \cos\vartheta|\eta_8\rangle - \sin\vartheta|\eta_0\rangle \\ |\eta'\rangle &= \sin\vartheta|\eta_8\rangle + \cos\vartheta|\eta_0\rangle. \end{aligned} \quad (31)$$

Eqs. (30) and (31) can be used to extract numerical values for $\bar{v}_3^{(1)}$ as a function of the mixing angle ϑ . To this end, we use the relation $B(\hat{m} - m_s) = m_\pi^2 - m_K^2$ which is valid at lowest chiral order, cf. Eq. (32). For $\vartheta = 0^\circ$ one obtains then $\bar{v}_3^{(1)} = -1.77 \times 10^{-3}$ GeV 2 , whereas $\vartheta = -20^\circ$ [11] leads to $\bar{v}_3^{(1)} = -0.76 \times 10^{-3}$ GeV 2 . The relation between the η - η' mixing angle ϑ and $\bar{v}_3^{(1)}$ in terms of the meson masses will change, of course, at higher chiral orders. We will therefore not treat $\bar{v}_3^{(1)}$ as a function of ϑ and leave its numerical value undetermined in the present investigation.

Up to second chiral order the masses for the pseudoscalar mesons read (with

a mixing angle $\vartheta = 0^\circ$)

$$\begin{aligned}\overset{\circ}{m}_\pi^2 &= 2B\hat{m} \\ \overset{\circ}{m}_K^2 &= B[\hat{m} + m_s] \\ \overset{\circ}{m}_\eta^2 &= \frac{2}{3}B[\hat{m} + 2m_s] \\ \overset{\circ}{m}_{\eta'}^2 &= \frac{2}{F^2}\bar{v}_0^{(2)} - \frac{8}{F^2}B[2\hat{m} + m_s]\left(\bar{v}_2^{(2)} - \frac{F^2}{12} - \sqrt{\frac{2}{3}}\bar{v}_3^{(1)}\right)\end{aligned}\quad (32)$$

with $\overset{\circ}{m}_P$ denoting the leading terms in the expansion of the masses. Note the $\bar{v}_0^{(2)}$ term in the expression for $\overset{\circ}{m}_{\eta'}$ which does not vanish in the chiral limit, i.e. the η' is not a Goldstone boson. The mixing angle ϑ is given by

$$\tan 2\vartheta = \frac{2m_{08}^2}{\overset{\circ}{m}_{\eta'}^2 - \overset{\circ}{m}_\eta^2}. \quad (33)$$

It can be expanded in powers of $1/\overset{\circ}{m}_{\eta'}^2$ so that one arrives at

$$\tan 2\vartheta = \frac{2m_{08}^2}{\overset{\circ}{m}_{\eta'}^2} + \mathcal{O}(p^4). \quad (34)$$

In this framework the generalized Gell-Mann-Okubo mass relation for the pseudoscalar mesons reads

$$\sin^2 \vartheta \overset{\circ}{m}_{\eta'}^2 + \cos^2 \vartheta \overset{\circ}{m}_\eta^2 = \frac{1}{3}(4\overset{\circ}{m}_K^2 - \overset{\circ}{m}_\pi^2) \quad (35)$$

which reduces to the conventional formula if one uses from Eq. (34) $\vartheta \sim \mathcal{O}(p^2)$, i.e. $\cos \vartheta \sim 1$ and $\sin \vartheta \sim 0$. For the physical values of the meson masses the above relation (35) yields $|\vartheta| \approx 10^\circ$. We have now established the theory at lowest order and can proceed by calculating the loop contributions in the following section.

4 Masses and decay constants

We have set up the effective Lagrangian for the calculation of the masses and decay constants of the pseudoscalar meson nonet up to fourth chiral order, i.e. one-loop order. The next step is to calculate the contributions from chiral loops. Employing dimensional regularization for the loop integrals amounts to a chiral expansion in m_P^2/Λ_χ^2 , $P = \pi, K, \eta, \eta'$. While the expansion parameters are small for the Goldstone boson octet, it is close to unity for the η' with $m_{\eta'}/\Lambda_\chi \sim 0.8$, and the convergence of the series is doubtful. The mass of the η' is a quantity of zeroth chiral order, $m_{\eta'} \sim \mathcal{O}(p^0)$, and this will spoil the chiral counting scheme:

higher loop graphs will also contribute to lower chiral orders. The situation is thus similar to baryon chiral perturbation theory where the nucleon mass sets a scale of similar size.

In [8] a new regularization method has been introduced, the so-called infrared regularization, which preserves the chiral counting scheme in the presence of massive fields, while keeping Lorentz and chiral invariance explicit at all stages. This method has so far only been employed in baryon chiral perturbation theory, see e.g. [8, 12, 13], but is applicable for any massive particle. In this section, we present the calculation of the masses and decay constants of the meson nonet up to one-loop order in infrared regularization. We will compare our results with the expressions obtained in dimensional regularization. At one-loop order only the tadpole graph contributes. The fundamental loop integral is given by

$$I = \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m_P^2 + i\epsilon} \quad (36)$$

with m_P being the mass of the meson inside the loop. We use the physical mass which is consistent to the order we are working. According to [8] the integral can be decomposed into a regular and a singular part, $I = S + R$, where both the pieces S and R preserve the symmetries of the Lagrangian. The regular part is a polynomial in the quark masses and can thus be absorbed by a suitable redefinition of the LECs. We will drop the regular part of this integral and keep the singular components which are given by

$$\begin{aligned} S_\phi &= m_\phi^2 \left[2L + \frac{1}{16\pi^2} \ln \frac{m_\phi^2}{\mu^2} \right] \quad \phi = \pi, K, \eta \\ S_{\eta'} &= 0 \end{aligned} \quad (37)$$

with

$$L = \frac{\mu^{d-4}}{16\pi^2} \left\{ \frac{1}{d-4} - \frac{1}{2} [\ln 4\pi + 1 - \gamma_E] \right\}. \quad (38)$$

and $\gamma_E = 0.5772\dots$ is the Euler-Mascheroni constant. The quantity μ is the scale introduced in the regularization of the integral. The divergent pieces of the chiral invariant singular parts constitute polynomials in the quark masses and can be cancelled by an appropriate renormalization of the coupling constants. Our main concern in this introductory paper is the applicability of $U(3)$ chiral perturbation theory to phenomenology, in particular the convergence of the chiral series. The more technical issue of the renormalization prescription for the LECs β_i of $\mathcal{L}^{(4)}$ will be neglected. (The complete renormalization of the one-loop functional in dimensional regularization has been given in [4].) We will therefore assume that both the regular part and the divergences from the singular part have been absorbed by a redefinition of the LECs and will use the same notation for the

renormalized coupling constants. In our calculation this amounts to keeping only the chiral logarithms of the loops with the Goldstone bosons,

$$S_\phi \rightarrow \frac{1}{16\pi^2} m_\phi^2 \ln \frac{m_\phi^2}{\mu^2} \quad \phi = \pi, K, \eta. \quad (39)$$

The calculation of the masses for the meson nonet up to fourth chiral order including one-loop corrections from $\mathcal{L}^{(0+2)}$ then yields

$$m_P^2 = \overset{\circ}{m}_P^2 + \frac{1}{F^2} C_P^{ab}(\mu) m_a^2 m_b^2 + \frac{1}{16\pi^2 F^2} D_P^{ab} m_a^2 m_b^2 \ln \frac{m_b^2}{\mu^2} + \Delta_P(\mu) \quad (40)$$

with $P = \pi, K, \eta, \eta'$. The counterterms contained in C_P^{ab} cancel the μ -dependence of the chiral logarithms and Δ_P includes both the corrections to $\overset{\circ}{m}_{\eta, \eta'}^2$ due to η - η' mixing and the tadpole contribution of the $\bar{v}_0^{(4)}$ vertex. The explicit expressions for C_P^{ab} , D_P^{ab} and Δ_P can be found in App. B.

The generalization to the corresponding expressions in dimensional regularization is straightforward. It is exclusively the tadpole contribution of the η' which adds to the nonanalytic pieces, so that for the masses $m_P^{2(dim)}$ in dimensional regularization we can write

$$m_P^{2(dim)} = m_P^2 + \frac{1}{16\pi^2 F^2} D_P^{an'} m_a^2 m_{\eta'}^2 \ln \frac{m_{\eta'}^2}{\mu^2} + \Delta_P^{(dim)}(\mu) \quad (41)$$

with m_P^2 given in Eq. (40). The $D_P^{an'}$ and $\Delta_P^{(dim)}$ are given in App. B. Of course, the difference between both schemes vanishes for $\mu = m_{\eta'}$ which is a peculiarity of the one-loop calculation of the masses since only the tadpole contributes. The degeneracy of both regularization schemes for $\mu = m_{\eta'}$ disappears in a two-loop calculation or for other processes which yield different nonanalytic contributions. An investigation of this is beyond the scope of the present work. Here, we evaluate the difference of both regularization schemes by varying the scale μ between the ρ mass, $m_\rho = 770$ MeV, and $\Lambda_\chi \sim 1.2$ GeV. Furthermore, the values of most of the couplings are not known and have to be determined in principle from experiment. In order to obtain numerical results we set all the couplings equal to zero except those from standard chiral perturbation theory, i.e. $\bar{v}_1^{(0)} = \bar{v}_2^{(0)} = F^2/4$ at second chiral order and the kinetic term of the η_0 has been normalized as explained above by using $\bar{v}_1^{(0)}/3 - \bar{v}_4^{(0)} = F^2/12$. At fourth chiral order we use the central phenomenological values for the renormalized $\beta_i^{(0)}(m_\rho)$ as given in [15]. These are in units of 10^{-3} : $\bar{\beta}_4^{(0)} = -0.3$, $\bar{\beta}_5^{(0)} = 1.4$, $\bar{\beta}_6^{(0)} = -0.2$, $\bar{\beta}_7^{(0)} = -0.4$ and $\bar{\beta}_8^{(0)} = 0.9$. The couplings $\bar{\beta}_{17}^{(0)}$ and $\bar{\beta}_{18}^{(0)}$ are OZI-violating corrections, while $\bar{\beta}_{25,26,52}^{(1)}$ are parity violating operators. Resonance exchange calculations as performed in [16] yield vanishing values for $\bar{\beta}_{17}^{(0)} \dots \bar{\beta}_{52}^{(1)}$ since the resonance couplings used within this approach obey both the OZI-rule and are parity conserving. We will therefore

set the values of these counterterms equal to zero. Table 1 shows the dependence of the next-to-leading order mass contributions on the scale μ for the values $\mu = m_\rho = 770$ MeV, $\mu = m_{\eta'} = 958$ MeV and $\mu = \Lambda_\chi = 1.2$ GeV, where we have used an η - η' mixing angle of both $\vartheta = 0^\circ$ and $\vartheta = -20^\circ$ [11]. In the calculation of the higher chiral orders we have kept for convenience the $\bar{\beta}_i$ at the scale $\mu = m_\rho$, $\bar{\beta}_i(m_\rho)$. Then the dependence of the meson masses on the renormalization scales stems only from the chiral logarithms and is rather weak. This in turn implies that the scale dependence of the $\bar{\beta}_i$ which should compensate this effect is also rather weak. We also present the numerical dependence of the masses on the unknown couplings $\bar{v}_i^{(j)}$ not known from standard chiral perturbation theory in Table 2. This table should be read as follows: e.g. the fourth order contribution to the η' mass is given as (in units of GeV^2 and $\vartheta = -20^\circ$, $\mu = m_\rho = 0.77$ GeV)

$$\begin{aligned} m_{\eta'}^2 - \overset{\circ}{m}_{\eta'}^2 &= -0.05601 - 21.47\bar{v}_0^{(4)} + 90.15\bar{v}_1^{(2)} - 58.41\bar{v}_2^{(2)} + 21.80\bar{v}_2^{(4)} \\ &\quad + 148.22\bar{v}_3^{(1)} + 8.73\bar{v}_3^{(3)} - 26.13\bar{v}_4^{(2)} \end{aligned} \quad (42)$$

in infrared regularization and

$$\begin{aligned} m_{\eta'}^2 - \overset{\circ}{m}_{\eta'}^2 &= -0.11904 + 296.14\bar{v}_0^{(4)} - 129.92\bar{v}_1^{(2)} + 173.76\bar{v}_2^{(2)} - 300.69\bar{v}_2^{(4)} \\ &\quad + 53.31\bar{v}_3^{(1)} + 152.20\bar{v}_3^{(3)} + 556.86\bar{v}_4^{(2)} \end{aligned} \quad (43)$$

in dimensional regularization. The first number denotes the fourth order contribution from the couplings of standard chiral perturbation theory as given in Table 1. From these results it becomes obvious that the nonanalytic pieces of the η' loops proportional to the unknown couplings $\bar{v}_i^{(j)}$ are numerically much more significant in dimensional regularization than in infrared regularization. Using dimensional regularization will eventually lead to a breakdown of the chiral expansion. Since in this scheme higher loops with more η' -propagators correspond to higher powers in $m_{\eta'}^2$, we cannot expect the chiral series to converge. This can be prevented by using infrared regularization similar to the situation with nucleons in baryon chiral perturbation theory.

We now turn to the calculation of the pseudoscalar decay constants. They are defined by

$$\langle 0 | A_\mu^a | P \rangle = i p_\mu F_P^a \quad P = \pi, K, \eta, \eta' \quad (44)$$

with $A_\mu^a = \bar{q} \gamma_\mu \gamma_5 \frac{1}{2} \lambda_a q$ and $\langle \lambda^a \lambda^b \rangle = 2\delta^{ab}$. One introduces the parametrization

$$\begin{aligned} F_\eta^8 &= \cos \vartheta_8 F_8, & F_\eta^0 &= -\sin \vartheta_0 F_0, \\ F_{\eta'}^8 &= \sin \vartheta_8 F_8, & F_{\eta'}^0 &= \cos \vartheta_0 F_0, \end{aligned} \quad (45)$$

which upon inversion leads to

$$(F_8)^2 = (F_\eta^8)^2 + (F_{\eta'}^8)^2, \quad (F_0)^2 = (F_\eta^0)^2 + (F_{\eta'}^0)^2. \quad (46)$$

The results for the decay constants can be written in the form

$$F_\phi = F \left(1 + \frac{1}{F^2} G_\phi^a(\mu) m_a^2 + \frac{1}{16\pi^2 F^2} H_\phi^a m_a^2 \ln \frac{m_a^2}{\mu^2} \right) \quad (47)$$

with $\phi = \pi, K, 8$ and the counterterms contained in G_ϕ^a have absorbed the divergences from the loops. The coefficients G and H can be found in App. C together with the pertinent Z -factors. Due to the appearance of many unknown LECs we will proceed similar to the case of the masses by separating the known chiral logarithms and coupling constants as given from standard chiral perturbation theory from the undetermined couplings $\bar{v}_i^{(j)}$. Table 3 shows in analogy to Table 2 the dependence of the pseudoscalar decay constants on the unknown parameters $\bar{v}_i^{(j)}$ where we have chosen $\mu = m_\rho$ and $\vartheta = -20^\circ$. For F_8 , e.g., we obtain (in units of GeV)

$$F_8 = F + 0.0444 - 0.3769\bar{v}_1^{(2)} - 0.2336\bar{v}_4^{(2)} \quad (48)$$

in infrared regularization whereas the pertinent result in dimensional regularization reads

$$F_8 = F + 0.0444 + 5.1983\bar{v}_1^{(2)} + 3.2217\bar{v}_4^{(2)} \quad (49)$$

with the first number being the contribution from the known LECs of standard chiral perturbation theory. Again, the η' loops lead to significant contributions in dimensional regularization.

The numerical value for F_0 cannot be determined from experiment since it depends on the running scale of QCD. The result for F_0 may be written as

$$F_0 = \sqrt{6\lambda} \left(1 + \frac{1}{F^2} G_0^a(\mu) m_a^2 + \frac{1}{16\pi^2 F^2} H_0^a m_a^2 \ln \frac{m_a^2}{\mu^2} \right) \equiv \frac{\sqrt{6\lambda}}{F} \bar{F}_0 \quad (50)$$

where \bar{F}_0 is scale invariant. The pertinent coefficients are given in App. C and the higher order contributions for \bar{F}_0 are shown in Table 3. They read in units of GeV

$$\bar{F}_0 = F - 0.0060 - 0.3769\bar{v}_1^{(2)} + 1.3643\bar{v}_4^{(2)} \quad (51)$$

in infrared regularization and

$$\bar{F}_0 = F - 0.0060 + 5.1983\bar{v}_1^{(2)} - 18.8266\bar{v}_4^{(2)}. \quad (52)$$

in dimensional regularization. The first number includes the couplings known from standard chiral perturbation theory with all the remaining parameters set equal to zero.

Finally, we would like to comment on the numerical values of the angles ϑ_0 and ϑ_8 . Their exact values up to one-loop order cannot be extracted since

some of the couplings are unknown. We will therefore proceed in analogy to the masses and decay constants by using the phenomenological values for the LECs of standard chiral perturbation theory and by neglecting the remaining ones. Using the identities

$$\tan \vartheta_8 = \frac{F_{\eta'}^8}{F_\eta^8}, \quad \tan \vartheta_0 = -\frac{F_\eta^0}{F_{\eta'}^0} \quad (53)$$

we obtain $\vartheta_0 = -4.42^\circ$ and $\vartheta_8 = -30.01^\circ$ for an η - η' mixing angle of -20° . In order to obtain an estimate of the uncertainty that results from using different values of the η - η' mixing angle, we perform the same calculation by employing $\vartheta = -13^\circ$ as given in [17]. The pertinent angles ϑ_0 and ϑ_8 are then $\vartheta_0 = 3.26^\circ$ and $\vartheta_8 = -24.06^\circ$.

5 Summary

In this investigation, we have presented an effective field theory which describes the interactions of the Goldstone boson octet with the corresponding singlet η' without imposing $1/N_c$ counting rules. The method has been illustrated by calculating the masses and decay constants of the pseudoscalar meson nonet up to one-loop order. The relevant effective Lagrangian up to fourth chiral order has been given. It turns out – as already discussed in [3] – that the LECs and the singlet field itself depend on the running scale of QCD. Rescaling the singlet field however yields QCD scale invariant coupling constants and the only scale dependent quantity of the effective theory shows up as a prefactor of the singlet axial current. This is in complete agreement with QCD since the axial vector current has anomalous dimension and acquires multiplicative renormalization.

Since the mass of the η' , $m_{\eta'}$, is close to the scale of chiral symmetry breaking, Λ_χ , dimensional regularization is not well suited for performing the loop integration. It yields an expansion in $m_{\eta'}/\Lambda_\chi$, thus causing the breakdown of the chiral expansion. This can be prevented by using so-called infrared regularization which suppresses the η' contribution to the amplitudes [8]. In the present work, the nonanalytic pieces of the one-loop integrals have been compared between both regularization schemes and it has been found that in dimensional regularization the η' tadpole leads to significant contributions, suggesting that the expansion in $m_{\eta'}$ will not be as well-behaved as for the Goldstone boson octet. In dimensional regularization one cannot expect higher loops to be less significant. A peculiarity of the calculation of the masses and decay constants is that the η' loop contributions vanish identically in infrared regularization, since at one-loop order only the tadpole contributes. This will change if one goes to higher loop order or considers other processes with different nonanalytic contributions. Nevertheless, the convergence of the chiral series will be improved by using infrared regularization as it is the case in baryon chiral perturbation theory. In order to confirm our

results other processes such as the hadronic decay modes of the η and η' will be investigated within this framework in future studies.

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A

In this Appendix, we present the dependence of the potentials β_i on the running scale of QCD. To this end, the Lagrangian $\mathcal{L}^{(4)}$ is rewritten in terms of $\hat{U}, \hat{a}_\mu, \nabla_\mu \psi$ ($\theta = 0$)

$$\begin{aligned}
\mathcal{L}^{(4)} = & (\beta_4 - i\beta_{22})e^{-\frac{i}{3}\psi}\langle\hat{\nabla}_\mu\hat{U}^\dagger\hat{\nabla}^\mu\hat{U}\rangle\langle\hat{U}^\dagger\chi\rangle + h.c. \\
& + (\beta_5 - i\beta_{21})e^{-\frac{i}{3}\psi}\langle\hat{\nabla}_\mu\hat{U}^\dagger\hat{\nabla}^\mu\hat{U}\hat{U}^\dagger\chi\rangle + h.c. \\
& + \frac{i}{3}(2\beta_5 + 3\beta_{18} - 2i\beta_{21} + 3i\beta_{23})e^{-\frac{i}{3}\psi}\nabla_\mu\psi\langle\hat{\nabla}^\mu\hat{U}^\dagger\chi\rangle + h.c. \\
& + \frac{1}{9}(3\beta_4 + \beta_5 - 9\beta_{17} + 3\beta_{18} - i\beta_{21} - 3i\beta_{22} + 3i\beta_{23} - 9i\beta_{24}) \\
& \quad \times e^{-\frac{i}{3}\psi}\nabla_\mu\psi\nabla^\mu\psi\langle\hat{U}^\dagger\chi\rangle + h.c. \\
& + (\beta_6 + \beta_7 + i\beta_{26})e^{-\frac{2i}{3}\psi}\langle\hat{U}^\dagger\chi\rangle^2 + h.c. \\
& + 2(\beta_6 - \beta_7)\langle\hat{U}^\dagger\chi\rangle\langle\chi^\dagger\hat{U}\rangle + \beta_{12}\langle\chi^\dagger\chi\rangle \\
& + (\beta_8 - i\beta_{25})e^{\frac{2i}{3}\psi}\langle\chi^\dagger\hat{U}\chi^\dagger\hat{U}\rangle + h.c. \\
& - (\beta_{52} - i\beta_{53})e^{-\frac{i}{3}\psi}\partial_\mu\nabla^\mu\theta\langle\hat{U}^\dagger\chi\rangle + h.c.
\end{aligned} \tag{A.1}$$

where we have also added the operators $\mathcal{O}_i, i = 21, 22, 23, 24, 53$. Although they do not contribute to the masses and decay constants, they are needed to reveal the scale dependence of the potentials. In order for the effective Lagrangian to remain invariant under a change of the QCD scale, the potentials β_i have to transform as

$$\begin{aligned}
(\beta_4 - i\beta_{22})(x) \rightarrow & (\beta_4 - i\beta_{22})(Z_A x)e^{\frac{i}{3}(1-Z_A)x} \\
(\beta_5 - i\beta_{21})(x) \rightarrow & (\beta_5 - i\beta_{21})(Z_A x)e^{\frac{i}{3}(1-Z_A)x} \\
(2\beta_5 + 3\beta_{18} - 2i\beta_{21} + 3i\beta_{23})(x) \rightarrow & (2\beta_5 + 3\beta_{18} - 2i\beta_{21} + 3i\beta_{23})(Z_A x) \\
& \times e^{\frac{i}{3}(1-Z_A)x}Z_A \\
(3\beta_4 + \beta_5 - 9\beta_{17} + 3\beta_{18} - i\beta_{21} \\
- 3i\beta_{22} + 3i\beta_{23} - 9i\beta_{24})(x) \rightarrow & (3\beta_4 + \beta_5 - 9\beta_{17} + 3\beta_{18} - i\beta_{21} \\
& - 3i\beta_{22} + 3i\beta_{23} - 9i\beta_{24})(Z_A x) \\
& \times e^{\frac{i}{3}(1-Z_A)x}Z_A^2 \\
(\beta_6 + \beta_7 + i\beta_{26})(x) \rightarrow & (\beta_6 + \beta_7 + i\beta_{26})(Z_A x)e^{\frac{2i}{3}(1-Z_A)x} \\
(\beta_6 - \beta_7)(x) \rightarrow & (\beta_6 - \beta_7)(Z_A x) \\
\beta_{12}(x) \rightarrow & \beta_{12}(Z_A x) \\
(\beta_8 + i\beta_{25})(x) \rightarrow & (\beta_8 + i\beta_{25})(Z_A x)e^{\frac{2i}{3}(1-Z_A)x} \\
(\beta_{52} - i\beta_{53})(x) \rightarrow & (\beta_{52} - i\beta_{53})(Z_A x)e^{\frac{i}{3}(1-Z_A)x}Z_A.
\end{aligned} \tag{A.2}$$

Note that the term $\beta_{12}\langle\chi\chi^\dagger\rangle$ contains a contact term, $\beta_{12}(0)\langle\chi\chi^\dagger\rangle$, which involves the renormalization of the corresponding counterterm in QCD, $\beta\langle ss^\dagger\rangle$. Such a term is consistent with the symmetries of QCD and is needed to render the effective action in QCD finite when the cutoff is removed. This contact term of the QCD Lagrangian is absorbed in the coupling constant $\beta_{12}(0)$ of the effective theory. The renormalization of the coupling constant $\beta_{12}(0)$ thus involves the renormalization factors relevant for β and is not covered by the above renormalization prescription of the potential β_{12} . Since the contact term does not contribute here, we can safely neglect this complication. Rescaling the singlet field as advocated in Sec. 3 modifies the potentials according to

$$\begin{aligned}
(\bar{\beta}_4 - i\bar{\beta}_{22})(x) &= (\beta_4 - i\beta_{22})\left(\frac{F}{\sqrt{\lambda}}x\right)e^{\frac{i}{3}(\sqrt{6}-\frac{F}{\sqrt{\lambda}})x} \\
(\bar{\beta}_5 - i\bar{\beta}_{21})(x) &= (\beta_5 - i\beta_{21})\left(\frac{F}{\sqrt{\lambda}}x\right)e^{\frac{i}{3}(\sqrt{6}-\frac{F}{\sqrt{\lambda}})x} \\
(2\bar{\beta}_5 + 3\bar{\beta}_{18} - 2i\bar{\beta}_{21} + 3i\bar{\beta}_{23})(x) &= (2\beta_5 + 3\beta_{18} - 2i\beta_{21} + 3i\beta_{23})\left(\frac{F}{\sqrt{\lambda}}x\right) \\
&\quad \times e^{\frac{i}{3}(\sqrt{6}-\frac{F}{\sqrt{\lambda}})x} \frac{F}{\sqrt{6\lambda}} \\
(3\bar{\beta}_4 + \bar{\beta}_5 - 9\bar{\beta}_{17} + 3\bar{\beta}_{18} - i\bar{\beta}_{21} \\
- 3i\bar{\beta}_{22} + 3i\bar{\beta}_{23} - 9i\bar{\beta}_{24})(x) &= (3\beta_4 + \beta_5 - 9\beta_{17} + 3\beta_{18} - i\beta_{21} \\
&\quad - 3i\beta_{22} + 3i\beta_{23} - 9i\beta_{24})\left(\frac{F}{\sqrt{\lambda}}x\right) \\
&\quad \times e^{\frac{i}{3}(\sqrt{6}-\frac{F}{\sqrt{\lambda}})x} \frac{F^2}{6\lambda} \\
(\bar{\beta}_6 + \bar{\beta}_7 + i\bar{\beta}_{26})(x) &= (\beta_6 + \beta_7 + i\beta_{26})\left(\frac{F}{\sqrt{\lambda}}x\right)e^{\frac{2i}{3}(\sqrt{6}-\frac{F}{\sqrt{\lambda}})x} \\
(\bar{\beta}_6 - \bar{\beta}_7)(x) &= (\beta_6 - \beta_7)\left(\frac{F}{\sqrt{\lambda}}x\right) \\
\bar{\beta}_{12}(x) &= \beta_{12}\left(\frac{F}{\sqrt{\lambda}}x\right) \\
(\bar{\beta}_8 + i\bar{\beta}_{25})(x) &= (\beta_8 + i\beta_{25})\left(\frac{F}{\sqrt{\lambda}}x\right)e^{\frac{2i}{3}(\sqrt{6}-\frac{F}{\sqrt{\lambda}})x} \\
(\bar{\beta}_{52} - i\bar{\beta}_{53})(x) &= (\beta_{52} - i\beta_{53})\left(\frac{F}{\sqrt{\lambda}}x\right)e^{\frac{i}{3}(\sqrt{6}-\frac{F}{\sqrt{\lambda}})x} \frac{F}{\sqrt{6\lambda}}. \tag{A.3}
\end{aligned}$$

The $\bar{\beta}_i$ are the QCD scale invariant potentials which we use in our calculation, and they are expanded in η_0

$$\begin{aligned}
\bar{\beta}_i\left(\frac{\eta_0}{F}\right) &= \bar{\beta}_i^{(0)} + \bar{\beta}_i^{(2)}\frac{\eta_0^2}{F^2} + \bar{\beta}_i^{(4)}\frac{\eta_0^4}{F^4} + \dots \quad \text{for } i = 4, 5, 6, 7, 8, 12, 17, 18, 53 \\
\bar{\beta}_i\left(\frac{\eta_0}{F}\right) &= \bar{\beta}_i^{(1)}\frac{\eta_0}{F} + \bar{\beta}_i^{(3)}\frac{\eta_0^3}{F^3} + \dots \quad \text{for } i = 21, 22, 23, 24, 25, 26, 52. \tag{A.4}
\end{aligned}$$

B

In this Appendix, we list the coefficients C_P^{ab} , D_P^{ab} and Δ_P from Eqs. (40) and (41).

$$\begin{aligned}
C_\pi^{\pi\pi} &= -8(\bar{\beta}_4^{(0)} + \bar{\beta}_5^{(0)} - 2(\bar{\beta}_6^{(0)} + \bar{\beta}_8^{(0)})) \\
C_\pi^{\pi K} = C_\pi^{K\pi} &= -8(\bar{\beta}_4^{(0)} - 2\bar{\beta}_6^{(0)}) \\
C_K^{\pi K} = C_K^{K\pi} &= -4(\bar{\beta}_4^{(0)} - 2\bar{\beta}_6^{(0)}) \\
C_K^{KK} &= -8(2\bar{\beta}_4^{(0)} + \bar{\beta}_5^{(0)} - 4\bar{\beta}_6^{(0)} - 2\bar{\beta}_8^{(0)}) \\
C_\eta^{\pi\pi} &= -\frac{2}{3}(8(\bar{\beta}_6^{(0)} - 8\bar{\beta}_7^{(0)} - 3\bar{\beta}_8^{(0)}) \cos^2 \vartheta \\
&\quad + 8(2\sqrt{2}(\bar{\beta}_6^{(0)} + \bar{\beta}_7^{(0)}) - \sqrt{3}\bar{\beta}_{26}^{(1)}) \sin(2\vartheta) \\
&\quad - (4(2\bar{\beta}_6^{(0)} - 3\bar{\beta}_6^{(2)} + 2\bar{\beta}_7^{(0)} + 6\bar{\beta}_8^{(0)}) \\
&\quad - 9(2\bar{\beta}_8^{(2)} + \bar{\beta}_{12}^{(2)}) - 4\sqrt{6}(3\bar{\beta}_{25}^{(1)} + \bar{\beta}_{26}^{(1)})) \sin^2 \vartheta) \\
C_\eta^{\pi K} = C_\eta^{K\pi} &= \frac{4}{3}(4(\bar{\beta}_6^{(0)} - 4(2\bar{\beta}_7^{(0)} + \bar{\beta}_8^{(0)})) \cos^2 \vartheta \\
&\quad - 2(2\sqrt{2}(\bar{\beta}_6^{(0)} + \bar{\beta}_7^{(0)} + 2\bar{\beta}_8^{(0)}) - \sqrt{3}(2\bar{\beta}_{25}^{(1)} + \bar{\beta}_{26}^{(1)})) \sin(2\vartheta) \\
&\quad + (4(2\bar{\beta}_6^{(0)} - 3\bar{\beta}_6^{(2)} + 2\bar{\beta}_7^{(0)} - 2\bar{\beta}_8^{(0)}) \\
&\quad + 3(2\bar{\beta}_8^{(2)} + \bar{\beta}_{12}^{(2)}) + 4\sqrt{6}(\bar{\beta}_{25}^{(1)} - \bar{\beta}_{26}^{(1)})) \sin^2 \vartheta) \\
C_\eta^{\pi\eta} = C_\eta^{\eta\pi} &= -\frac{2}{3}(3(2\bar{\beta}_4^{(0)} - 3\bar{\beta}_{17}^{(0)} + \bar{\beta}_{18}^{(0)}) - (2\bar{\beta}_5^{(0)} - 9\bar{\beta}_{17}^{(0)} + 3\bar{\beta}_{18}^{(0)}) \cos(2\vartheta) \\
&\quad - 2\sqrt{2}(2\bar{\beta}_5^{(0)} + 3\bar{\beta}_{18}^{(0)}) \sin(2\vartheta)) \\
C_\eta^{KK} &= \frac{8}{3}(16(\bar{\beta}_6^{(0)} + \bar{\beta}_7^{(0)} + \bar{\beta}_8^{(0)}) \cos^2 \vartheta \\
&\quad + 4(2\sqrt{2}(\bar{\beta}_6^{(0)} + \bar{\beta}_7^{(0)} + \bar{\beta}_8^{(0)}) - \sqrt{3}(\bar{\beta}_{25}^{(1)} + \bar{\beta}_{26}^{(1)})) \sin(2\vartheta) \\
&\quad + (4(2\bar{\beta}_6^{(0)} - 3\bar{\beta}_6^{(2)} + 2\bar{\beta}_7^{(0)} + 2\bar{\beta}_8^{(0)}) \\
&\quad - 3(2\bar{\beta}_8^{(2)} + \bar{\beta}_{12}^{(2)}) - 4\sqrt{6}(\bar{\beta}_{25}^{(1)} + \bar{\beta}_{26}^{(1)})) \sin^2 \vartheta) \\
C_\eta^{K\eta} = C_\eta^{\eta K} &= -\frac{4}{3}(3(2\bar{\beta}_4^{(0)} + \bar{\beta}_5^{(0)} - 3\bar{\beta}_{17}^{(0)} + \bar{\beta}_{18}^{(0)}) + (\bar{\beta}_5^{(0)} + 9\bar{\beta}_{17}^{(0)} - 3\bar{\beta}_{18}^{(0)}) \cos(2\vartheta) \\
&\quad + \sqrt{2}(2\bar{\beta}_5^{(0)} + 3\bar{\beta}_{18}^{(0)}) \sin(2\vartheta)) \\
C_{\eta'}^{\pi\pi} &= -\frac{2}{3}(8(\bar{\beta}_6^{(0)} - 8\bar{\beta}_7^{(0)} - 3\bar{\beta}_8^{(0)}) \sin^2 \vartheta \\
&\quad - 8(2\sqrt{2}(\bar{\beta}_6^{(0)} + \bar{\beta}_7^{(0)}) - \sqrt{3}\bar{\beta}_{26}^{(1)}) \sin(2\vartheta) \\
&\quad - (4(2\bar{\beta}_6^{(0)} - 3\bar{\beta}_6^{(2)} + 2\bar{\beta}_7^{(0)} + 6\bar{\beta}_8^{(0)}) \\
&\quad - 9(2\bar{\beta}_8^{(2)} + \bar{\beta}_{12}^{(2)}) - 4\sqrt{6}(3\bar{\beta}_{25}^{(1)} + \bar{\beta}_{26}^{(1)})) \cos^2 \vartheta)
\end{aligned}$$

$$\begin{aligned}
C_{\eta'}^{\pi K} &= C_{\eta'}^{K\pi} = \frac{4}{3}(4(\bar{\beta}_6^{(0)} - 4(2\bar{\beta}_7^{(0)} + \bar{\beta}_8^{(0)})) \sin^2 \vartheta \\
&\quad + 2(2\sqrt{2}(\bar{\beta}_6^{(0)} + \bar{\beta}_7^{(0)} + 2\bar{\beta}_8^{(0)}) - \sqrt{3}(2\bar{\beta}_{25}^{(1)} + \bar{\beta}_{26}^{(1)})) \sin(2\vartheta) \\
&\quad + (4(2\bar{\beta}_6^{(0)} - 3\bar{\beta}_6^{(2)} + 2\bar{\beta}_7^{(0)} - 2\bar{\beta}_8^{(0)}) + 3(2\bar{\beta}_8^{(2)} + \bar{\beta}_{12}^{(2)}) \\
&\quad \quad + 4\sqrt{6}(\bar{\beta}_{25}^{(1)} - \bar{\beta}_{26}^{(1)})) \cos^2 \vartheta \\
C_{\eta'}^{\pi\eta'} &= C_{\eta'}^{\eta'\pi} = -\frac{2}{3}(3(2\bar{\beta}_4^{(0)} - 3\bar{\beta}_{17}^{(0)} + \bar{\beta}_{18}^{(0)}) + (2\bar{\beta}_5^{(0)} - 9\bar{\beta}_{17}^{(0)} + 3\bar{\beta}_{18}^{(0)}) \cos(2\vartheta) \\
&\quad + 2\sqrt{2}(2\bar{\beta}_5^{(0)} + 3\bar{\beta}_{18}^{(0)}) \sin(2\vartheta)) \\
C_{\eta'}^{KK} &= \frac{8}{3}(16(\bar{\beta}_6^{(0)} + \bar{\beta}_7^{(0)} + \bar{\beta}_8^{(0)}) \sin^2 \vartheta \\
&\quad - 4(2\sqrt{2}(\bar{\beta}_6^{(0)} + \bar{\beta}_7^{(0)} + \bar{\beta}_8^{(0)}) - \sqrt{3}(\bar{\beta}_{25}^{(1)} + \bar{\beta}_{26}^{(1)})) \sin(2\vartheta) \\
&\quad + (4(2\bar{\beta}_6^{(0)} - 3\bar{\beta}_6^{(2)} + 2\bar{\beta}_7^{(0)} + 2\bar{\beta}_8^{(0)}) \\
&\quad \quad - 3(2\bar{\beta}_8^{(2)} + \bar{\beta}_{12}^{(2)}) - 4\sqrt{6}(\bar{\beta}_{25}^{(1)} + \bar{\beta}_{26}^{(1)})) \cos^2 \vartheta) \\
C_{\eta'}^{K\eta'} &= C_{\eta'}^{\eta'K} = -\frac{4}{3}(3(2\bar{\beta}_4^{(0)} + \bar{\beta}_5^{(0)} - 3\bar{\beta}_{17}^{(0)} + \bar{\beta}_{18}^{(0)}) - \sqrt{2}(2\bar{\beta}_5^{(0)} + 3\bar{\beta}_{18}^{(0)}) \sin(2\vartheta) \\
&\quad - (\bar{\beta}_5^{(0)} + 9\bar{\beta}_{17}^{(0)} - 3\bar{\beta}_{18}^{(0)}) \cos(2\vartheta)). \tag{B.1}
\end{aligned}$$

$$\begin{aligned}
D_{\pi}^{\pi\pi} &= \frac{1}{2} \\
D_{\pi}^{\pi\eta} &= -\frac{1}{6F^2}(F^2 \cos^2 \vartheta - (\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \sin(2\vartheta) \\
&\quad + 2(F^2 + 12(\bar{v}_1^{(2)} - \bar{v}_2^{(2)}) + 4\sqrt{6}\bar{v}_3^{(1)}) \sin^2 \vartheta) \\
D_{\pi}^{\pi\eta'} &= -\frac{1}{6F^2}(F^2 \sin^2 \vartheta + (\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \sin(2\vartheta) \\
&\quad + 2(F^2 + 12(\bar{v}_1^{(2)} - \bar{v}_2^{(2)}) + 4\sqrt{6}\bar{v}_3^{(1)}) \cos^2 \vartheta) \\
D_K^{\pi\eta} &= \frac{\cos \vartheta}{12F^2}(F^2 \cos \vartheta + 2(\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \sin \vartheta) \\
D_K^{\pi\eta'} &= \frac{\sin \vartheta}{12F^2}(F^2 \sin \vartheta - 2(\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \cos \vartheta) \\
D_K^{K\eta} &= -\frac{\sin \vartheta}{3F^2}((F^2 + 4(3(\bar{v}_1^{(2)} - \bar{v}_2^{(2)}) + \sqrt{6}\bar{v}_3^{(1)})) \sin \vartheta \\
&\quad + (\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \cos \vartheta) \\
D_K^{K\eta'} &= -\frac{\cos \vartheta}{3F^2}((F^2 + 4(3(\bar{v}_1^{(2)} - \bar{v}_2^{(2)}) + \sqrt{6}\bar{v}_3^{(1)})) \cos \vartheta \\
&\quad - (\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \sin \vartheta)
\end{aligned}$$

$$\begin{aligned}
D_K^{\eta\eta} &= \frac{\cos^2 \vartheta}{4} \\
D_K^{\eta'\eta'} &= \frac{\sin^2 \vartheta}{4} \\
D_\eta^{\pi\pi} &= -\frac{1}{2F^2}(F^2 \cos^2 \vartheta - (\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \sin(2\vartheta) \\
&\quad + 2(F^2 + 12(\bar{v}_1^{(2)} - \bar{v}_2^{(2)}) + 4\sqrt{6}\bar{v}_3^{(1)}) \sin^2 \vartheta) \\
D_\eta^{\pi K} &= \frac{\cos \vartheta}{3F^2}(F^2 \cos \vartheta + 2(\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \sin \vartheta) \\
D_\eta^{\pi\eta} &= \frac{1}{18F^2}(7F^2 \cos^4 \vartheta + 2(5\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \cos^2 \vartheta \sin(2\vartheta) \\
&\quad + 3(F^2 + 4\sqrt{6}\bar{v}_3^{(1)} - 12\bar{v}_2^{(2)}) \sin^2(2\vartheta) \\
&\quad + 8(\sqrt{2}(F^2 - 36\bar{v}_2^{(2)}) + 12\sqrt{3}(\bar{v}_3^{(1)} - 3\bar{v}_3^{(3)})) \sin^2 \vartheta \sin(2\vartheta) \\
&\quad - 2(F^2 - 72(\bar{v}_2^{(2)} - 3\bar{v}_2^{(4)}) + 8\sqrt{6}(\bar{v}_3^{(1)} - 9\bar{v}_3^{(3)})) \sin^4 \vartheta) \\
D_\eta^{\pi\eta'} &= D_{\eta'}^{\pi\eta} = \frac{1}{144F^2}(3(3F^2 + 32\bar{v}_2^{(2)} - 144\bar{v}_2^{(4)} + 48\sqrt{6}\bar{v}_3^{(3)}) \\
&\quad + (7F^2 - 16(18\bar{v}_2^{(2)} - 27\bar{v}_2^{(4)} - \sqrt{6}(4\bar{v}_3^{(1)} - 9\bar{v}_3^{(3)}))) \cos(4\vartheta) \\
&\quad - 4(\sqrt{2}(F^2 + 144\bar{v}_2^{(2)}) - 4\sqrt{3}(7\bar{v}_3^{(1)} - 36\bar{v}_3^{(3)})) \sin(4\vartheta)) \\
D_\eta^{KK} &= -\frac{4 \sin \vartheta}{3F^2}((F^2 + 4(3(\bar{v}_1^{(2)} - \bar{v}_2^{(2)}) + \sqrt{6}\bar{v}_3^{(1)})) \sin \vartheta \\
&\quad + (\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \cos \vartheta) \\
D_\eta^{K\eta} &= -\frac{2}{9F^2}(4F^2 \cos^4 \vartheta + 4(\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \cos^2 \vartheta \sin(2\vartheta) \\
&\quad + 3(F^2 + 4\sqrt{6}\bar{v}_3^{(1)} - 12\bar{v}_2^{(2)}) \sin^2(2\vartheta) \\
&\quad + 2(\sqrt{2}(F^2 - 36\bar{v}_2^{(2)}) + 12\sqrt{3}(\bar{v}_3^{(1)} - 3\bar{v}_3^{(3)})) \sin^2 \vartheta \sin(2\vartheta) \\
&\quad + (F^2 - 8(9(\bar{v}_2^{(2)} - 3\bar{v}_2^{(4)}) - \sqrt{6}(\bar{v}_3^{(1)} - 9\bar{v}_3^{(3)}))) \sin^4 \vartheta) \\
D_\eta^{K\eta'} &= D_{\eta'}^{K\eta} = -\frac{1}{36F^2}(3(3F^2 - 8(5\bar{v}_2^{(2)} - 9\bar{v}_2^{(4)} - \sqrt{6}(\bar{v}_3^{(1)} - 3\bar{v}_3^{(3)}))) \\
&\quad + (7F^2 - 8(9\bar{v}_2^{(2)} + 27\bar{v}_2^{(4)} - \sqrt{6}(5\bar{v}_3^{(1)} + 9\bar{v}_3^{(3)}))) \cos(4\vartheta) \\
&\quad - 4(\sqrt{2}(F^2 + 36\bar{v}_2^{(2)}) - 4\sqrt{3}(\bar{v}_3^{(1)} - 9\bar{v}_3^{(3)})) \sin(4\vartheta)) \\
D_\eta^{\eta K} &= \cos^2 \vartheta \\
D_\eta^{\eta\eta} &= -\frac{4 \sin^2 \vartheta}{F^2}(2\bar{v}_1^{(2)} - 3\bar{v}_4^{(2)} + 3\bar{v}_4^{(2)} \cos(2\vartheta)) \\
D_\eta^{\eta\eta'} &= D_{\eta'}^{\eta\eta} = -\frac{2 \cos^2 \vartheta}{F^2}(2\bar{v}_1^{(2)} - 3\bar{v}_4^{(2)} + 3\bar{v}_4^{(2)} \cos(2\vartheta)) \\
D_\eta^{\eta'\eta'} &= D_{\eta'}^{\eta'\eta} = -\frac{2 \sin^2 \vartheta}{F^2}(2\bar{v}_1^{(2)} - 3\bar{v}_4^{(2)} - 3\bar{v}_4^{(2)} \cos(2\vartheta))
\end{aligned}$$

$$\begin{aligned}
D_{\eta'}^{\pi\pi} &= -\frac{1}{2F^2}(F^2 \sin^2 \vartheta + (\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \sin(2\vartheta) \\
&\quad + 2(F^2 + 4(3\bar{v}_1^{(2)} - 3\bar{v}_2^{(2)} + \sqrt{6}\bar{v}_3^{(1)})) \cos^2 \vartheta) \\
D_{\eta'}^{\pi K} &= \frac{\sin \vartheta}{3F^2}(F^2 \sin \vartheta - 2(\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \cos \vartheta) \\
D_{\eta'}^{\pi\eta'} &= \frac{1}{18F^2}(7F^2 \sin^4 \vartheta - 2(5\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \sin^2 \vartheta \sin(2\vartheta) \\
&\quad + 3(F^2 + 4\sqrt{3}\bar{v}_3^{(1)} - 12\bar{v}_2^{(2)}) \sin^2(2\vartheta) \\
&\quad - 8(\sqrt{2}(F^2 - 36\bar{v}_2^{(2)}) + 12\sqrt{3}(\bar{v}_3^{(1)} + 3\bar{v}_3^{(3)})) \cos^2 \vartheta \sin(2\vartheta) \\
&\quad - 2(F^2 - 8(9\bar{v}_2^{(2)} - 27\bar{v}_2^{(4)} - \sqrt{6}(\bar{v}_3^{(1)} - 9\bar{v}_3^{(3)}))) \cos^4 \vartheta) \\
D_{\eta'}^{KK} &= -\frac{4 \cos \vartheta}{3F^2}((F^2 + 4(3\bar{v}_1^{(2)} - 3\bar{v}_2^{(2)} + \sqrt{6}\bar{v}_3^{(1)})) \cos \vartheta \\
&\quad - (\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \sin \vartheta) \\
D_{\eta'}^{K\eta'} &= -\frac{1}{9F^2}(8F^2 \sin^4 \vartheta - 8(\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)}) \sin^2 \vartheta \sin(2\vartheta) \\
&\quad + 6(F^2 + 4\sqrt{6}\bar{v}_3^{(1)} - 12\bar{v}_2^{(2)}) \sin^2(2\vartheta) \\
&\quad - 4(\sqrt{2}(F^2 - 36\bar{v}_2^{(2)}) + 12\sqrt{3}(\bar{v}_3^{(1)} + 3\bar{v}_3^{(3)})) \cos^2 \vartheta \sin(2\vartheta) \\
&\quad + 2(F^2 - 8(9\bar{v}_2^{(2)} - 27\bar{v}_2^{(4)} - \sqrt{6}(\bar{v}_3^{(1)} - 9\bar{v}_3^{(3)}))) \cos^4 \vartheta) \\
D_{\eta'}^{\eta' K} &= \sin^2 \vartheta \\
D_{\eta'}^{\eta'\eta'} &= -\frac{4 \cos^2 \vartheta}{F^2}(2\bar{v}_1^{(2)} - 3\bar{v}_4^{(2)} - 3\bar{v}_4^{(2)} \cos(2\vartheta)). \tag{B.2}
\end{aligned}$$

$$\begin{aligned}
\Delta_\eta &= (\overset{\circ}{m}_{\eta'}^2 - \overset{\circ}{m}_\eta^2) \sin^2 \vartheta + (\overset{\circ}{m}_K^2 - \overset{\circ}{m}_\pi^2) \frac{2(\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)})}{3F^2} \sin(2\vartheta) \\
&\quad + 12\bar{v}_0^{(4)} \sin^4 \vartheta \frac{1}{16\pi^2 F^4} m_\eta^2 \ln \frac{m_\eta^2}{\mu^2} \\
\Delta_{\eta'} &= (\overset{\circ}{m}_\eta^2 - \overset{\circ}{m}_{\eta'}^2) \sin^2 \vartheta - (\overset{\circ}{m}_K^2 - \overset{\circ}{m}_\pi^2) \frac{2(\sqrt{2}F^2 + 4\sqrt{3}\bar{v}_3^{(1)})}{3F^2} \sin(2\vartheta) \\
&\quad + 3\bar{v}_0^{(4)} \sin^2(2\vartheta) \frac{1}{16\pi^2 F^4} m_\eta^2 \ln \frac{m_\eta^2}{\mu^2} \\
\Delta_\eta^{(dim)} &= 3\bar{v}_0^{(4)} \sin^2(2\vartheta) \frac{1}{16\pi^2 F^4} m_{\eta'}^2 \ln \frac{m_{\eta'}^2}{\mu^2} \\
\Delta_{\eta'}^{(dim)} &= 12\bar{v}_0^{(4)} \cos^4 \vartheta \frac{1}{16\pi^2 F^4} m_{\eta'}^2 \ln \frac{m_{\eta'}^2}{\mu^2} \tag{B.3}
\end{aligned}$$

C

The coefficients G_ϕ^a and H_ϕ^a of Eqs. (47) and (50) read

$$\begin{aligned}
G_\pi^\pi &= 4(\bar{\beta}_4^{(0)} + \bar{\beta}_5^{(0)}) \\
G_\pi^K &= 8\bar{\beta}_4^{(0)} \\
G_K^\pi &= 4\bar{\beta}_4^{(0)} \\
G_K^K &= 8\bar{\beta}_4^{(0)} + 4\bar{\beta}_4^{(0)} \\
G_8^\pi &= \frac{1}{3}(3(4\bar{\beta}_4^{(0)} - 2\bar{\beta}_5^{(0)} + 3\bar{\beta}_{17}^{(0)} - \bar{\beta}_{18}^{(0)}) \\
&\quad + (2\bar{\beta}_5^{(0)} - 9\bar{\beta}_{17}^{(0)} + 3\bar{\beta}_{18}^{(0)}) \cos(4\vartheta) \\
&\quad + 2\sqrt{2}(2\bar{\beta}_5^{(0)} + 3\bar{\beta}_{18}^{(0)}) \sin(4\vartheta)) \\
G_8^K &= \frac{2}{3}(3(4\bar{\beta}_4^{(0)} + 3\bar{\beta}_5^{(0)} + 3\bar{\beta}_{17}^{(0)} - \bar{\beta}_{18}^{(0)}) \\
&\quad - (\bar{\beta}_5^{(0)} + 9\bar{\beta}_{17}^{(0)} - 3\bar{\beta}_{18}^{(0)}) \cos(4\vartheta) \\
&\quad - \sqrt{2}(2\bar{\beta}_5^{(0)} + 3\bar{\beta}_{18}^{(0)}) \sin(4\vartheta)) \\
G_0^\pi &= \frac{1}{3}((12\bar{\beta}_4^{(0)} + 6\bar{\beta}_5^{(0)} - 45\bar{\beta}_{17}^{(0)} + 15\bar{\beta}_{18}^{(0)} - 6\sqrt{6}\bar{\beta}_{52}^{(1)}) \\
&\quad - (2\bar{\beta}_5^{(0)} - 9\bar{\beta}_{17}^{(0)} + 3\bar{\beta}_{18}^{(0)}) \cos(4\vartheta) \\
&\quad - 2\sqrt{2}(2\bar{\beta}_5^{(0)} + 3\bar{\beta}_{18}^{(0)}) \sin(4\vartheta)) \\
G_0^K &= \frac{2}{3}(3(4\bar{\beta}_4^{(0)} + \bar{\beta}_5^{(0)} - 15\bar{\beta}_{17}^{(0)} + 5\bar{\beta}_{18}^{(0)} - 2\sqrt{6}\bar{\beta}_{52}^{(1)}) \\
&\quad + (\bar{\beta}_5^{(0)} + 9\bar{\beta}_{17}^{(0)} - 3\bar{\beta}_{18}^{(0)}) \cos(4\vartheta) \\
&\quad + \sqrt{2}(2\bar{\beta}_5^{(0)} + 3\bar{\beta}_{18}^{(0)}) \sin(4\vartheta)). \tag{C.1}
\end{aligned}$$

$$\begin{aligned}
H_\pi^\pi &= -1 \\
H_\pi^K &= -\frac{1}{2} \\
H_\pi^\eta &= \frac{2}{F^2} \bar{v}_1^{(2)} \sin^2 \vartheta \\
H_\pi^{\eta'} &= \frac{2}{F^2} \bar{v}_1^{(2)} \cos^2 \vartheta \\
H_K^\pi &= -\frac{3}{8} \\
H_K^K &= -\frac{3}{4} \\
H_K^\eta &= -\frac{1}{8F^2} (3F^2 \cos^2 \vartheta - 16\bar{v}_1^{(2)} \sin^2 \vartheta) \\
H_K^{\eta'} &= -\frac{1}{8F^2} (3F^2 \sin^2 \vartheta - 16\bar{v}_1^{(2)} \cos^2 \vartheta) \\
H_8^K &= -\frac{1}{8} (13 - \cos(4\vartheta)) \\
H_8^\eta &= \frac{\sin^2 \vartheta}{2F^2} (4\bar{v}_1^{(2)} + 3\bar{v}_4^{(2)} - 3\bar{v}_4^{(2)} \cos(4\vartheta)) \\
H_8^{\eta'} &= \frac{\cos^2 \vartheta}{2F^2} (4\bar{v}_1^{(2)} + 3\bar{v}_4^{(2)} - 3\bar{v}_4^{(2)} \cos(4\vartheta)) \\
H_0^K &= \frac{1}{4} \sin^2(2\vartheta) \\
H_0^\eta &= \frac{\sin^2 \vartheta}{2F^2} (4\bar{v}_1^{(2)} - 15\bar{v}_4^{(2)} + 3\bar{v}_4^{(2)} \cos(4\vartheta)) \\
H_0^{\eta'} &= \frac{\cos^2 \vartheta}{2F^2} (4\bar{v}_1^{(2)} - 15\bar{v}_4^{(2)} + 3\bar{v}_4^{(2)} \cos(4\vartheta)). \tag{C.2}
\end{aligned}$$

The Z -factors are given by

$$Z_P = 1 + \frac{1}{F^2} K_P^a m_a^2 + \frac{1}{16\pi^2 F^2} L_P^a m_a^2 \ln \frac{m_a^2}{\mu^2} \tag{C.3}$$

with the coefficients

$$\begin{aligned}
K_\pi^\pi &= -8(\bar{\beta}_4^{(0)} + \bar{\beta}_5^{(0)}) \\
K_\pi^K &= -16\bar{\beta}_4^{(0)} \\
K_K^\pi &= -8\bar{\beta}_4^{(0)} \\
K_K^K &= -8(2\bar{\beta}_4^{(0)} - \bar{\beta}_5^{(0)}) \\
K_\eta^\pi &= -\frac{4}{3}(6\bar{\beta}_4^{(0)} - 9\bar{\beta}_{17}^{(0)} + 3\bar{\beta}_{18}^{(0)} - (2\bar{\beta}_5^{(0)} - 9\bar{\beta}_{17}^{(0)} + 3\bar{\beta}_{18}^{(0)}) \cos(2\vartheta) \\
&\quad - 2\sqrt{2}(2\bar{\beta}_5^{(0)} + 3\bar{\beta}_{18}^{(0)}) \sin(2\vartheta)) \\
K_\eta^K &= -\frac{8}{3}(3(2\bar{\beta}_4^{(0)} + \bar{\beta}_5^{(0)} - 3\bar{\beta}_{17}^{(0)} + \bar{\beta}_{18}^{(0)}) + (\bar{\beta}_5^{(0)} + 9\bar{\beta}_{17}^{(0)} - 3\bar{\beta}_{18}^{(0)}) \cos(2\vartheta) \\
&\quad + \sqrt{2}(2\bar{\beta}_5^{(0)} + 3\bar{\beta}_{18}^{(0)}) \sin(2\vartheta)) \\
K_{\eta'}^\pi &= -\frac{4}{3}(6\bar{\beta}_4^{(0)} - 9\bar{\beta}_{17}^{(0)} + 3\bar{\beta}_{18}^{(0)} + (2\bar{\beta}_5^{(0)} - 9\bar{\beta}_{17}^{(0)} + 3\bar{\beta}_{18}^{(0)}) \cos(2\vartheta) \\
&\quad + 2\sqrt{2}(2\bar{\beta}_5^{(0)} + 3\bar{\beta}_{18}^{(0)}) \sin(2\vartheta)) \\
K_{\eta'}^K &= -\frac{8}{3}(3(2\bar{\beta}_4^{(0)} + \bar{\beta}_5^{(0)} - 3\bar{\beta}_{17}^{(0)} + \bar{\beta}_{18}^{(0)}) - (\bar{\beta}_5^{(0)} + 9\bar{\beta}_{17}^{(0)} - 3\bar{\beta}_{18}^{(0)}) \cos(2\vartheta) \\
&\quad - \sqrt{2}(2\bar{\beta}_5^{(0)} + 3\bar{\beta}_{18}^{(0)}) \sin(2\vartheta)). \tag{C.4}
\end{aligned}$$

$$\begin{aligned}
L_\pi^\pi &= \frac{2}{3} \\
L_\pi^K &= \frac{1}{3} \\
L_\pi^\eta &= -\frac{4}{F^2} \bar{v}_1^{(2)} \sin^2 \vartheta \\
L_\pi^{\eta'} &= -\frac{4}{F^2} \bar{v}_1^{(2)} \cos^2 \vartheta \\
L_K^\pi &= \frac{1}{4} \\
L_K^K &= \frac{1}{2} \\
L_K^\eta &= \frac{1}{4F^2} (F^2 \cos^2 \vartheta - 16\bar{v}_1^{(2)} \sin^2 \vartheta) \\
L_K^{\eta'} &= \frac{1}{4F^2} (F^2 \sin^2 \vartheta - 16\bar{v}_1^{(2)} \cos^2 \vartheta) \\
L_\eta^K &= \cos^2 \vartheta \\
L_\eta^\eta &= -\frac{2 \sin^2 \vartheta}{F^2} (2\bar{v}_1^{(2)} - 3\bar{v}_4^{(2)} + 3\bar{v}_4^{(2)} \cos(2\vartheta)) \\
L_\eta^{\eta'} &= -\frac{2 \cos^2 \vartheta}{F^2} (2\bar{v}_1^{(2)} - 3\bar{v}_4^{(2)} + 3\bar{v}_4^{(2)} \cos(2\vartheta)) \\
L_{\eta'}^K &= \sin^2 \vartheta \\
L_{\eta'}^\eta &= -\frac{2 \sin^2 \vartheta}{F^2} (2\bar{v}_1^{(2)} - 3\bar{v}_4^{(2)} - 3\bar{v}_4^{(2)} \cos(2\vartheta)) \\
L_{\eta'}^{\eta'} &= -\frac{2 \cos^2 \vartheta}{F^2} (2\bar{v}_1^{(2)} - 3\bar{v}_4^{(2)} - 3\bar{v}_4^{(2)} \cos(2\vartheta)). \tag{C.5}
\end{aligned}$$

References

- [1] J. Gasser, H. Leutwyler, Nucl. Phys. **B250** (1985) 465
- [2] H. Leutwyler, Phys. Lett. **B374** (1996) 163
- [3] R. Kaiser, H. Leutwyler, in Proc. Workshop “*Nonperturbative Methods in Quantum Field Theory*”, eds. A. W. Schreiber, A. G. Williams and A. W. Thomas (World Scientific, Singapore, 1998), hep-ph/9806336;
R. Kaiser, H. Leutwyler, “*Large N_c in chiral perturbation theory*”, hep-ph/0007101
- [4] P. Herrera-Siklody et al., Nucl. Phys. **B497** (1997) 345
- [5] B. Borasoy, Phys. Rev. **D61** (2000) 014011

- [6] See, e.g., V. Bernard, N. Kaiser, U.-G. Meißner, Int. J. Mod. Phys. **E4** (1995) 193
- [7] H.-B. Tang, “*A New Approach to Chiral Perturbation Theory for Matter Fields*”, hep-ph/9607436;
P. J. Ellis, H.-B. Tang, Phys. Rev. **C57** (1998) 3356
- [8] T. Becher, H. Leutwyler, Eur. Phys. J. **C9** (1999) 643
- [9] S. L. Adler, Phys. Rev. **177** (1969) 2426;
J. Kodaira, Nucl. Phys. **B165** (1980) 129
- [10] H. Leutwyler, Nucl. Phys. Proc. Suppl. **64** (1998) 223
- [11] Particle Data Group, C. Caso et al., Eur. Phys. J **C3** (1998) 1
- [12] P. J. Ellis, K. Torikoshi, Phys. Rev. **C61** (2000) 015205
- [13] M. Mojzis, J. Kambor, Phys. Lett. **B476** (2000) 344
- [14] P. Herrera-Siklódy et al., Phys. Lett. **B419** (1998) 326
- [15] J. Bijnens, G. Ecker, J. Gasser, “*Chiral Perturbation Theory*”, in 2nd DAPHNE Physics Handbook, INFN, Frascati, (1995), eds. L. Maiani, G. Pancheri, N. Paver, hep-ph/9411232
- [16] G. Ecker, J. Gasser, A. Pich, E. de Rafael, Nucl. Phys. **B321** (1989) 311
- [17] A. Bramon, R. Escribano, M. D. Scadron, Eur. Phys. J. **C7** (1999) 271

Table captions

Table 1 Shown are the next-to-leading order mass contributions in units of 10^{-3} GeV 2 both in dimensional and infrared regularization. For the scale μ of the chiral logarithms we used $\mu = m_\rho$, $\mu = m_{\eta'}$ and $\mu = \Lambda_\chi$ and employed the mixing angles $\vartheta = 0^\circ$ and $\vartheta = -20^\circ$. Only the couplings from standard chiral perturbation theory have been retained while neglecting the remaining LECs.

Table 2 The dependence of the next-to-leading order mass contributions on the unknown parameters $\bar{v}_i^{(j)}$ is given in units of GeV 2 . We used $\mu = m_\rho$ for the scale of the chiral logarithms and an η - η' mixing angle of $\vartheta = -20^\circ$.

Table 3 Given are the next-to-leading order contributions to the pseudoscalar decay constants in units of GeV both in infrared and in dimensional regularization. The first column of each regularization scheme shows the contribution which arises if only known LECs from standard chiral perturbation theory are kept whereas the second and third columns show the dependence on $\bar{v}_1^{(2)}$ and $\bar{v}_4^{(2)}$. We used $\mu = m_\rho$ for the scale of the chiral logarithms and an η - η' mixing angle of $\vartheta = -20^\circ$.

$m_P^2 - \overset{\circ}{m}_P^2$		$\vartheta = 0^\circ$				$\vartheta = -20^\circ$			
		π	K	η	η'	π	K	η	η'
ir	$\mu = m_\rho$	-0.79	-0.88	-13.52	23.53	-0.27	-3.43	-91.93	-56.01
	$\mu = m_{\eta'}$	-0.52	-8.19	-17.30	59.23	0.31	-12.38	-115.84	-20.84
	$\mu = \Lambda_\chi$	-0.26	-15.74	-21.19	96.03	0.90	-21.60	-140.49	15.42
dim	$\mu = m_\rho$	-2.69	-24.76	-44.73	6.97	-1.20	-27.00	-122.72	-119.04
	$\mu = m_{\eta'}$	-0.52	-8.19	-17.30	59.23	0.31	-12.38	-115.84	-20.84
	$\mu = \Lambda_\chi$	1.72	8.89	10.98	113.11	1.86	2.70	-108.74	80.41

Table 1

	$m_\pi^2 - \overset{\circ}{m}_\pi^2$		$m_K^2 - \overset{\circ}{m}_K^2$		$m_\eta^2 - \overset{\circ}{m}_\eta^2$		$m_{\eta'}^2 - \overset{\circ}{m}_{\eta'}^2$	
	ir	dim	ir	dim	ir	dim	ir	dim
$\propto \bar{v}_0^{(4)}$	0	0	0	0	-2.84	39.23	-21.47	296.14
$\propto \bar{v}_1^{(2)}$	0.16	-2.19	1.98	-27.28	13.38	-37.07	90.15	-129.92
$\propto \bar{v}_2^{(2)}$	-0.16	2.19	-1.98	27.28	-16.57	-28.31	-58.41	173.76
$\propto \bar{v}_2^{(4)}$	0	0	0	0	2.89	39.83	21.80	-300.69
$\propto \bar{v}_3^{(1)}$	0.38	-1.04	-1.40	-31.19	-88.47	-100.39	148.22	53.31
$\propto \bar{v}_3^{(3)}$	0	0	0	0	5.74	-11.36	8.73	152.20
$\propto \bar{v}_4^{(2)}$	0	0	0	0	-1.70	49.50	-26.13	556.86

Table 2

	infrared			dimensional		
	$\propto 1$	$\propto \bar{v}_1^{(2)}$	$\propto \bar{v}_4^{(2)}$	$\propto 1$	$\propto \bar{v}_1^{(2)}$	$\propto \bar{v}_4^{(2)}$
$F_\pi - F$	0.0066	-0.3769	0	0.0066	5.1983	0
$F_K - F$	0.0255	-0.3769	0	0.0243	5.1983	0
$F_8 - F$	0.0444	-0.3769	-0.2336	0.0444	5.1983	3.2217
$\bar{F}_0 - F$	-0.0060	-0.3769	1.3643	-0.0060	5.1983	-18.8266

Table 3